What is probability?

Probability is simply how likely something is to happen.

Example *If a coin is randomly flipped, how likely it lands on head or tail?*

Why we should learn probability?

- *Nowadays, randomness becomes prevalent in real life.*
  - *e.g., the revenue of apple store at APM next month is random*
  - *e.g., the length a patient with cancer can survive is random*
- *Probability is the fundamental **tool** for modelling, analyzing randomness*
  - *e.g., what is the most likely revenue of apple store at APM next month?*
  - *e.g., what is the most likely length a patient with cancer can survive?*
- *Probability is the fundamental **tool** for statistics, machine learning, etc.*

Chapter 1. Probability by Combinatorial Analysis

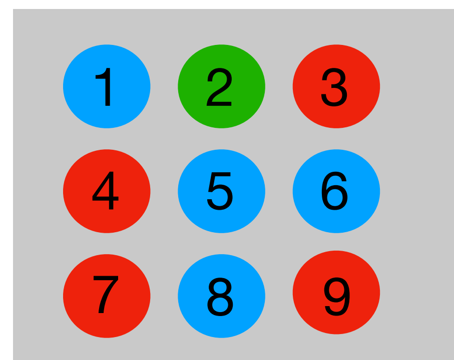
Outline

1.1 Introduction

Simple intuitive examples to understand probability

Example *If we randomly pick up a ball in the following box, what is the chance of selecting a red ball?*

we can view this chance as probability.



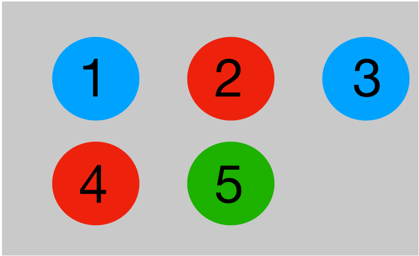
Solution *We see there are 9 balls in the box and 4 of them are red. Intuitively, we know the chance of selecting a red ball is 4/9.*

$$\frac{4}{9} = \frac{\text{4 outcomes are red balls } \{3,4,7,9\}}{\text{9 possible outcomes } \{1,2,3,4,5,6,7,8,9\}}$$

1.1 Introduction

Class Discussion

If we randomly pick up 2 balls in the following box, what is the chance that the 2 selected balls are both red?



Choose your answer?

Choice A

$\frac{1}{4}$

Choice B

$\frac{1}{5}$

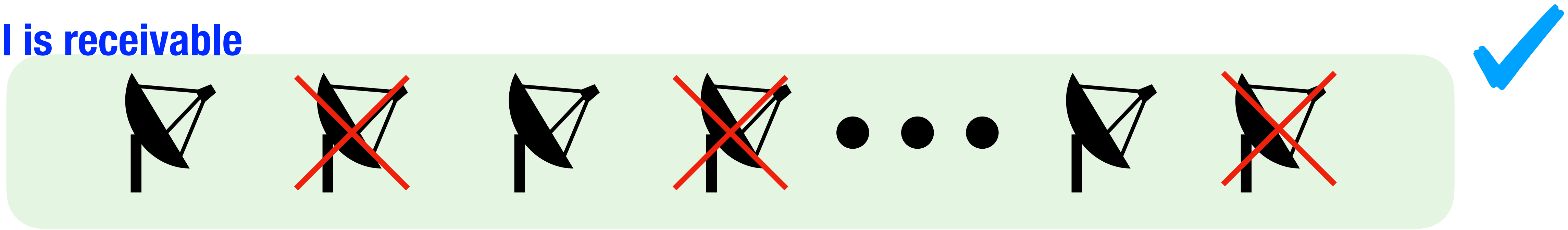
Choice C

$\frac{1}{10}$

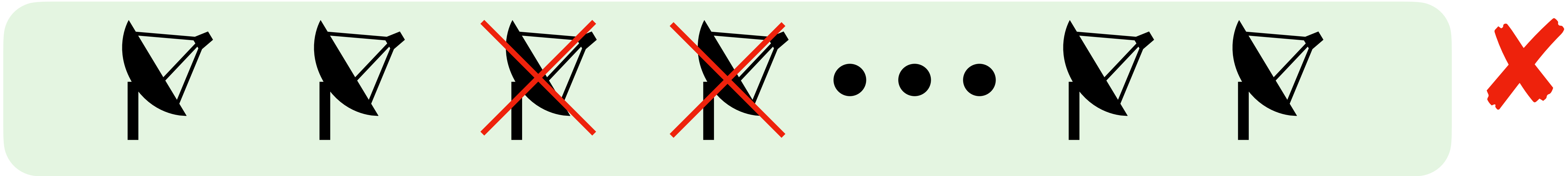
1.1 Introduction

Example A communication system is to consist of n identical antennas lined up in a linear order. The system can receive signals as long as no two consecutive antennas are defective. If it turns that exactly m of the n antennas are defective, what is the probability that the system can receive signals.

Signal is receivable



Signal is not receivable



1.1 Introduction

Example A communication system is to consist of n identical antennas lined up in a linear order. The system can receive signals as long as no two consecutive antennas are defective. If it turns that exactly m of the n antennas are defective, what is the probability that the system can receive signals.

Solution

Let us solve the problem in the special case where $n = 4$ and $m = 2$. In this case, there are 6 possible system configurations, namely,

$$\begin{matrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{matrix}$$

where 1 means that the antenna is working and 0 that it is defective. Note that the system can receive signals in the first 3 arrangements and fails in the remaining 3, it seems reasonable to take $\frac{3}{6} = \frac{1}{2}$ as the desired probability.

1.1 Introduction

Continue from last page…

For general n and m, we could regard the probability as

$$\textbf{Probability that system works} = \frac{\text{number of configurations that the system works well}}{\text{total number of all possible configurations}}$$

Why counting matters

From the above example, we see that it would be useful to have an effective method for counting the number of ways that things can occur. The mathematical theory of counting is formally known as *combinatorial analysis*

1.2 Principle of Counting

Concepts

Experiment

We use the concept "experiment" to denote a process whose outcome is random

Example of experiment

e.g. randomly pick a number from {10,100,1000,10000}

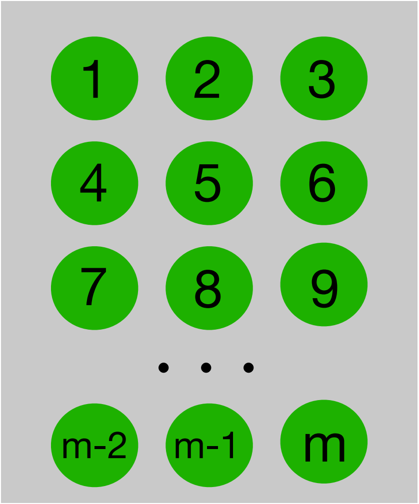
e.g. randomly toss a coin

e.g. randomly roll a die

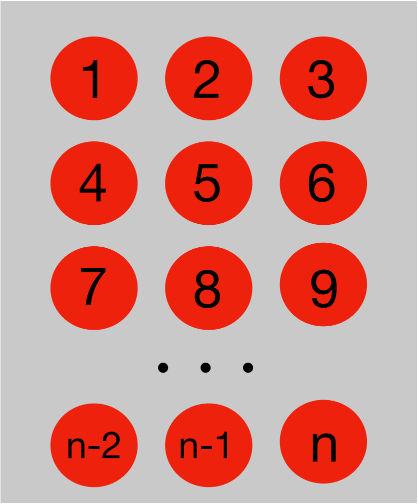
1.2 Principle of Counting

Theorem (The basic principle of counting). *Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if for each outcome of experiment a there are n possible outcomes of experiment 2, then together there are mn possible outcomes of the two experiments.*

Experiment 1



Experiment 2



1.2 Principle of Counting

Theorem (The basic principle of counting). *Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if for each outcome of experiment a there are n possible outcomes of experiment 2, then together there are mn possible outcomes of the two experiments.*

Proof We prove it by enumerating all the possible outcomes of the two experiments as follows:

$$\begin{array}{cccc} (1,1) & (1,2) & \cdots & (1,n) \\ (2,1) & (2,2) & \cdots & (2,n) \\ \vdots & \vdots & \vdots & \vdots \\ (m,1) & (m,2) & \cdots & (m,n) \end{array}$$

where we say that the outcome is (i, j) if experiment 1 results in its ith possible outcome and experiment 2 then results in the jth of its possible outcomes. Hence, the set of possible outcomes consists of m rows, each row containing n elements, which proves the result.

1.2 Principle of Counting

Example A small community consists of 10 women, each of whom has 3 children. If one woman and one of her children are to be chosen as mother and child of the year, how many different choices are possible?

Solution

By regarding the choice of the woman as the outcome of the first experiment and the subsequent choice of one of her children as the outcome of the second experiment, we see from the basic principle that there are $10 \times 3 = 30$ possible choices.

Example In a class of 40 students, we choose a president and a vice president. There are

$$40 \times 39 = 1560$$

possible choices.

1.2 Principle of Counting

Theorem (The generalized basic principle of counting). *If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes, and if for each of these n_1 possible outcomes there are n_2 possible outcomes of the second experiment, and if for each of the possible outcomes of the first two experiments there are n_3 possible outcomes of the third experiment, and if ... then there is a total of $n_1 n_2 \cdots n_r$ possible outcomes of the r experiments.*

Example How many different 7-place license plates are possible if the first 3 places for letters and the final 4 places by numbers?

Solution:
First 3 places each has 26 ways, and final 4 places each has 10 ways. Therefore, the total possible number of ways is

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 175,760,000.$$

1.2 Principle of Counting

Example How many different 7-place license plates are possible if the first 3 places for letters and the final 4 places by numbers and repetition among letters or numbers were prohibited?

Solution:
In this case, there would be $26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7 = 78,624,000$ possible license plates.

1.3 Permutations

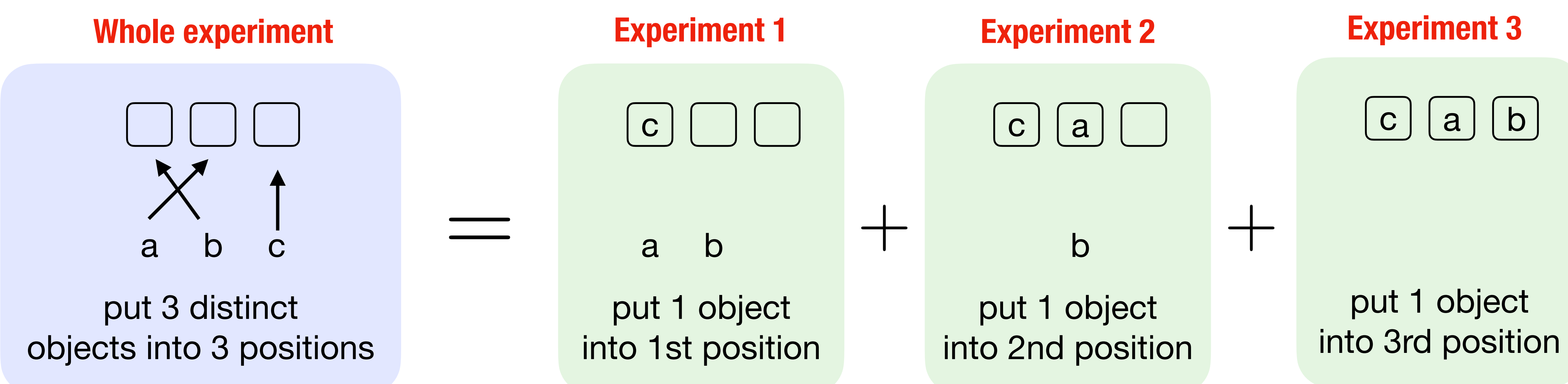
Definition

How many different *ordered arrangements* of the letters a, b, c are possible. By direct enumeration we see that there are 6:

$$abc \qquad acb \qquad bac \qquad bca \qquad cab \qquad cba$$

Each arrangement is known as a *permutation*.

For 3 objects, there are 6 possible permutations. This can be explained by the basic principle, since the first object in the permutation can be any of the 3, the second object in the permutation can then be chosen from any of the remaining 2, and the third object in the permutation is then chosen from the remaining one. Thus there are $3 \cdot 2 \cdot 1 = 6$ permutations.



1.3 Permutations

Theorem

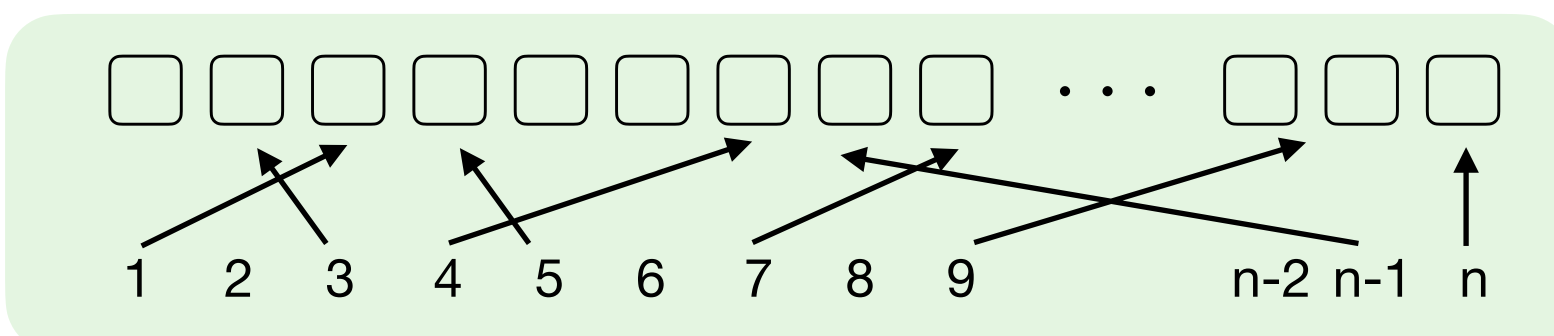
Suppose there are n (distinct) objects, then the total number of different arrangements is

$$n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n!$$

with the convention that

$$0! = 1.$$

Proof *The experiment is equivalent to arranging n distinct objects into n positions.*



To fill the 1st position, there are n choices.

To fill the 2nd position, there are n-1 choices.

To fill the 3rd position, there are n-2 choices.

. . .

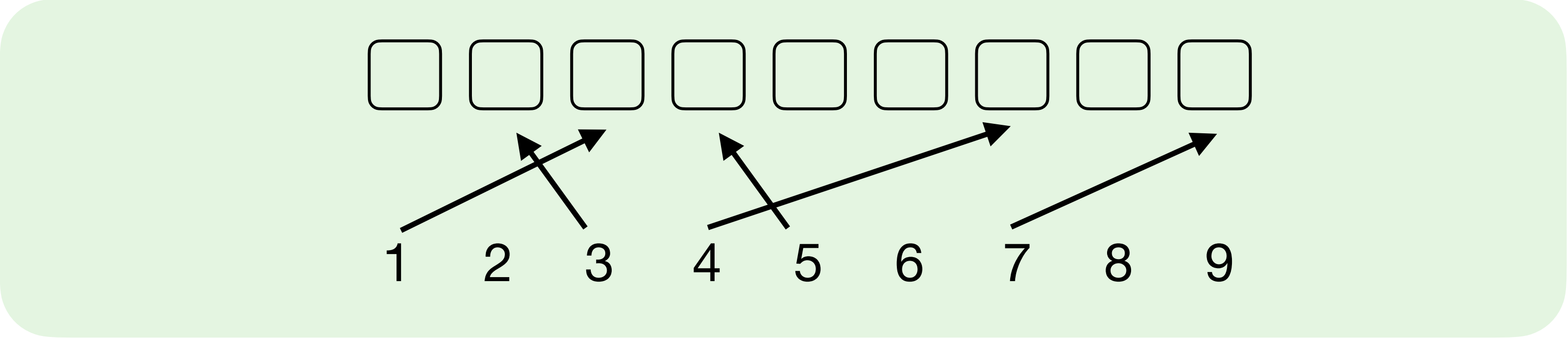
To fill the nth position, there are 1 choices.

In total

$$n \times (n-1) \times (n-2) \times \cdots \times 1 = n!$$

1.3 Permutations

Example Seating arrangement in a row: 9 people sitting in a row. There are $9! = 362,880$ ways.



1.3 Permutations

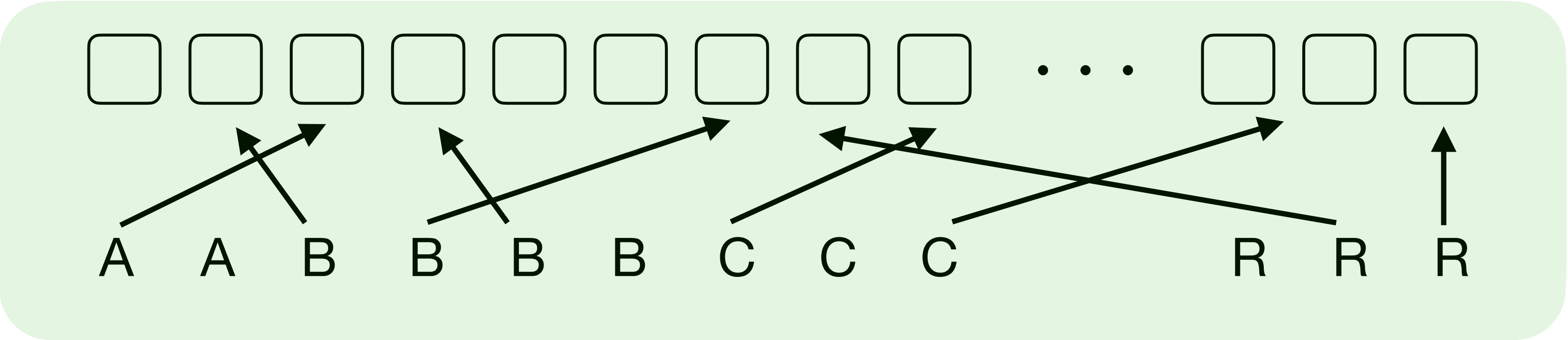
Theorem

For n objects of which n_1 are alike, n_2 are alike, ..., n_r are alike, there are

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

different permutations of the n objects.

Proof



- *First, we view these n objects as all distinct objects and permute them, there are n! permutations*
- *However, there are many objects are alike, so some permutations are essentially the same but counted more than once.*
- *For instance, consider the permutation ABBABBCCC……RRR*
- *When we permute all As, or all Bs, or all Cs, … or all Rs, essentially they are the same permutation. In other words, each permutation is counted* $n_1!n_2!\cdots n_r!$ *times.*

1.3 Permutations

Example How many ways to rearrange Mississippi?

$$n = 11$$

$$n_1 = 1 \qquad n_2 = 4 \qquad n_3 = 4 \qquad n_4 = 2$$

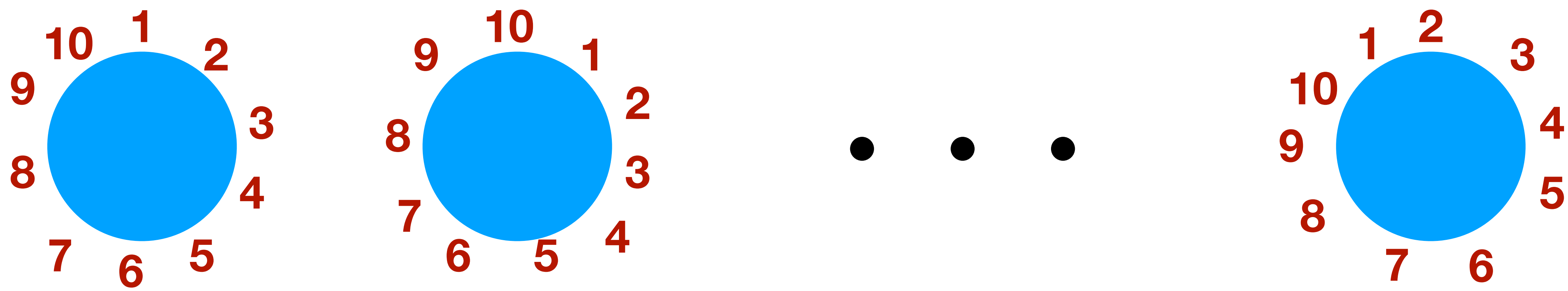
Solution:
11 letters of which 1 M, 4 I's, 4 S's and 2 P's.
There are

$$\frac{11!}{1!4!4!2!} = 34,650.$$

1.3 Permutations

Seating in circle

Example (Seating in circle). 10 people sitting around a round dining table. It is the relative positions that really matters – who is on your left, on your right. No. of seating arrangements is



Rotating it clock wisely gives the same arrangements

$$\frac{10!}{10} = 9!$$

1.3 Permutations

Seating in circle

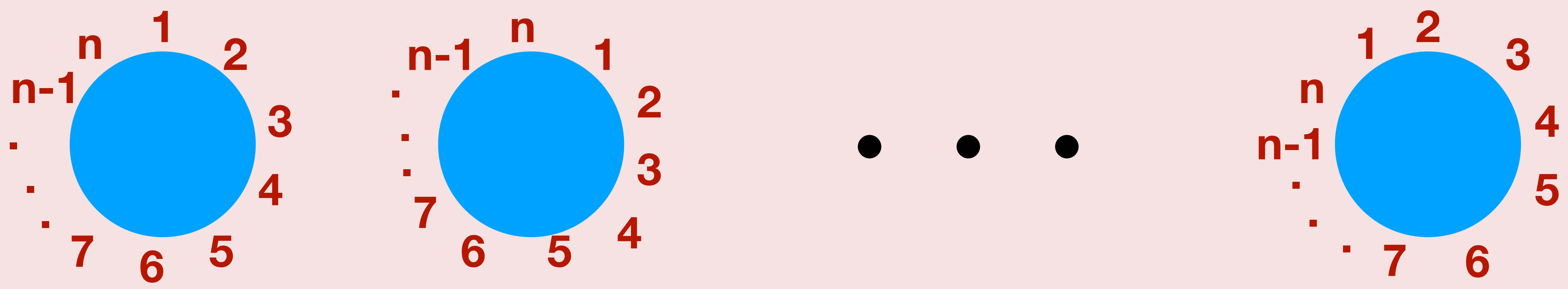
Theorem

Generally, for n people sitting in a circle, there are

$$\frac{n!}{n} = (n-1)!$$

possible arrangements.

Proof



Rotating it clock wisely
gives the same arrangements

1.3 Permutations

Seating in circle

Example (Making necklaces). n different pearls string in a necklace. Number of ways of stringing the pearls is

First, like arranging n objects in a round table, there are (n-1)! arrangements

In addition, for a necklace, we can flip it reversely, e.g., like a mirror. They are actually the same necklace

Mirror the necklaces gives the same one

Therefore, the answer is $\frac{(n-1)!}{2}$

1.4 Combinations

Example

In how many ways can we choose 3 items from 5 items: A, B, C, D and E?

5 ways to choose first item,
4 ways to choose second item, and
3 ways to choose third item
So the number of ways (in this order) is $5 \cdot 4 \cdot 3$.
However,

$$ABC, ACB, BAC, BCA, CAB \text{ and } CBA$$

will be considered as the same group.
So the number of different groups (order not important) is $\dfrac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1}$.

Basically, in combinations, we do not care about the order the items are chosen or arranged.

1.4 Combinations

Theorem

Generally, if there are n distinct objects, of which we choose a group of r items,

$$\text{Number of possible groups} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} \times \frac{(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}{(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}$$

$$= \frac{n!}{r!(n-r)!}.$$

denoted by → ${}_nC_r$ or $\binom{n}{r}$

called "n choose r"

Properties

For $r = 0, 1, \ldots, n$,

$$\binom{n}{r} = \binom{n}{n-r}.$$

$$\binom{n}{0} = \binom{n}{n} = 1.$$

Convention:
When n is a nonnegative integer, and $r < 0$ or $r > n$, take

$$\binom{n}{r} = 0.$$

1.4 Combinations

Example A committee of 3 is to be formed from a group of 20 people.

1. How many possible committees can be formed?

No. of different committees that can be formed = $\binom{20}{3} = 1140$.

2. Suppose further that, two guys: Peter and Paul refuse to serve in the same committee. How many possible committees can be formed with the restriction that these two guys don't serve together?

Two possible cases:

Case 1. Both of them are not in the committee.
Ways to do that $= \binom{18}{3} = 816$.

Case 2. One of them in.
Ways to form $= \binom{2}{1}\binom{18}{2} = 306$.

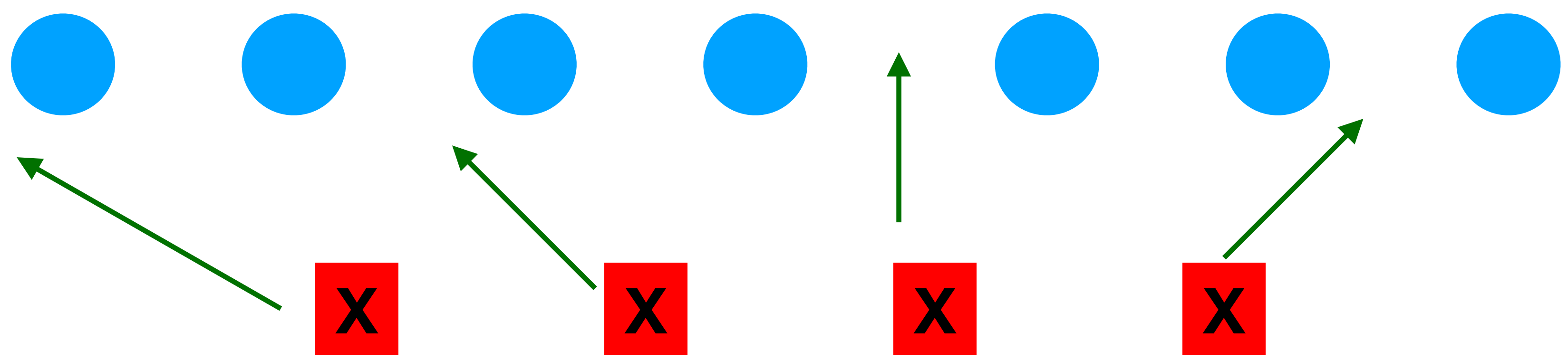
Total $= 816 + 306 = 1122$.

Alternative solution: (sketch)

$$\binom{20}{3} - \binom{2}{2}\binom{18}{1} = 1140 - 18 = 1122.$$

1.4 Combinations

Example Consider a set of n antennas of which m are defective and $n-m$ are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?



n-m functional antennas

m defective antennas

In total, there are n-m+1 slots to place the defective antennas. We need to choose m places to put the defective antennas.

$$\binom{n-m+1}{m}$$

1.4 Combinations

Useful Combinatorial Identities

Theorem

For $1 \le r \le n$,

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

Algebraic Proof

$$\text{RHS} = \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-r-1)!} = \frac{(n-1)!}{r!(n-r)!}[r+(n-r)] = \frac{n!}{r!(n-r)!}.$$

Combinatorial Proof

Consider the cases where the first object (i) is chosen, (ii) not chosen:

$$\binom{1}{1}\cdot\binom{n-1}{r-1} + \binom{1}{0}\cdot\binom{n-1}{r}.$$

1.4 Combinations

Useful Combinatorial Identities

Binomial Theorem

Let n be a nonnegative integer, then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

In view of the binomial theorem, $\binom{n}{k}$ is often referred to as the binomial coefficient.

Combinatorial Proof

1.4 Combinations

Useful Combinatorial Identities

Example How many subsets are there of a set consisting of n elements?

Since there are $\binom{n}{k}$ subsets of size k, the desired answer is

$$\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n.$$

This result could also have been obtained by considering whether each element in the set is being chosen or not (to be part of a subset). As there are 2^n possible assignments, the result follows.

Example

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Proof

Let $x = -1$, $y = 1$ in the binomial theorem.

1.4 Combinations

Useful Combinatorial Identities

Example

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots .$$

Proof

Start with the previous example. Move the negative terms to the right hand side of the equation.

1.5 Multinomial Coefficients

Example

A set of n distinct items is to be divided into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$, where $\sum_{i=1}^{r} n_i = n$. How many different divisions are possible?

Proof

Note that there are $\binom{n}{n_1}$ possible choices for the first group; for each choice of the first group, there are $\binom{n-n_1}{n_2}$ possible choices for the second group; for each choice of the first two groups, there are $\binom{n-n_1-n_2}{n_3}$ possible choices for the third group; and so on. It then follows from the generalized version of the basic counting principle that there are

$$\begin{aligned}
&\binom{n}{n_1} \times \binom{n-n_1}{n_2} \times \cdots \times \binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r} \\
&= \frac{n!}{(n-n_1)!n_1!} \times \frac{(n-n_1)!}{(n-n_1-n_2)!n_2!} \times \cdots \times \frac{(n-n_1-n_2-\cdots-n_{r-1})!}{0!n_r!} \\
&= \frac{n!}{n_1!n_2!\cdots n_r!}
\end{aligned}$$

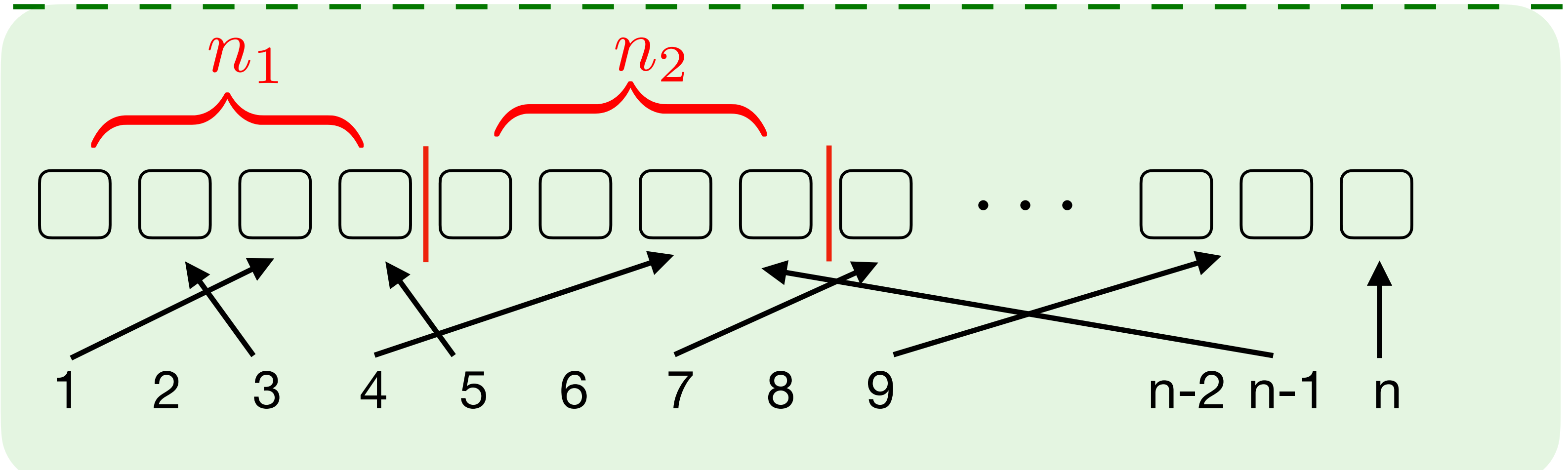
possible divisions.

1.5 Multinomial Coefficients

Example

A set of n distinct items is to be divided into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$, where $\sum_{i=1}^{r} n_i = n$. How many different divisions are possible?

Another Proof



The problem can be cast to arranging n objects in a line and then the first n_1 objects belong to 1st group; the second n_2 objects belong to the 2nd group; …..

There are n! different arrangements of n objects in a line.
However, for each arrangement, no matter how you permute the first n_1 objects, they are the same divisions; similarly, you can permute the second n_2 objects, they are also the same divisions;….

1.5 Multinomial Coefficients

Notations

If $n_1 + n_2 + \cdots + n_r = n$, we define $\binom{n}{n_1, n_2, \ldots, n_r}$ by

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}.$$

Example A police department in a small city consists of 10 officers. If the department policy is to have 5 of the officers patrolling the streets, 2 of the officers working full time at the station, and 3 of the officers on reserve at the station, how many different divisions of the 10 officers into the 3 groups are possible?

Solution:
There are $\frac{10!}{5!2!3!} = 2520$ possible divisions.

1.5 Multinomial Coefficients

Example Ten children are to be divided into an A team and a B team of 5 each. The A team will play in one league and the B team in another. How many different divisions are possible?

Solution:
There are $\frac{10!}{5!5!} = 252$ possible divisions.

Example In order to play a game of basketball, 10 children at a playground divide themselves into two teams of 5 each. How many different divisions are possible?

Note that this example is different from the previous one because now the order of the two teams is irrelevant. That is, there is no A and B team, but just a division consisting of 2 groups of 5 each. Hence, the desired answer is

$$\frac{10!/(5!5!)}{2!} = 126.$$

1.5 Multinomial Coefficients

Multinomial Theorem

$$(x_1+x_2+\cdots+x_r)^n = \sum_{(n_1,\ldots,n_r):n_1+\cdots+n_r=n} \binom{n}{n_1,n_2,\ldots,n_r} x_1^{n_1}x_2^{n_2}\cdots x_r^{n_r}$$

Note that the sum is over all nonnegative integer-valued vectors $(n_1,n_2,\ldots,n_r)$ such that $n_1+n_2+\cdots+n_r=n$.

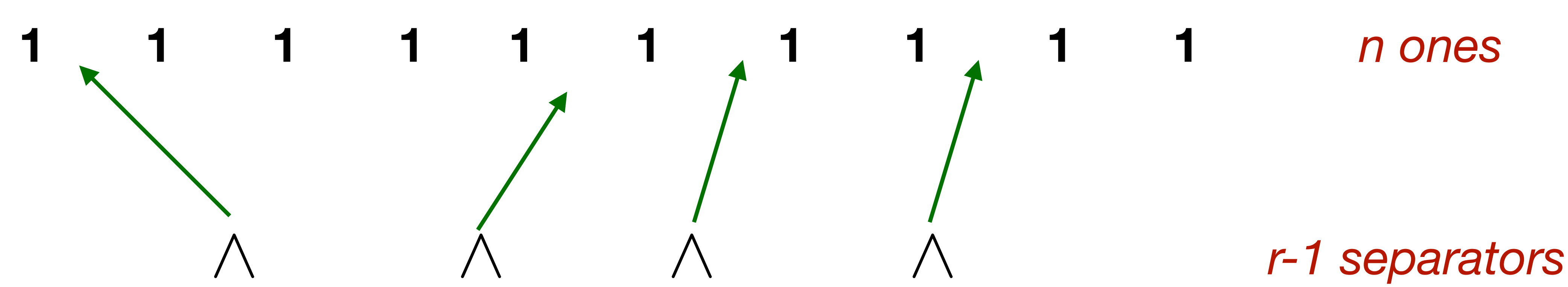
1.6 Number of Integer Solutions

Theorem *There are $\binom{n-1}{r-1}$ distinct positive integer-valued vectors $(x_1, x_2, \ldots, x_r)$ that satisfies the equation*

$$x_1 + x_2 + \cdots + x_r = n,$$

where $x_i > 0$ for $i = 1, \ldots, r$.

Proof



Choose r-1 from n-1 places to put the separators

1.6 Number of Integer Solutions

Theorem *There are $\binom{n+r-1}{r-1}$ distinct non-negative integer-valued vectors $(x_1, x_2, \ldots, x_r)$ that satisfies the equation*

$$x_1 + x_2 + \cdots + x_r = n,$$

where $x_i \geq 0$ for $i = 1, \ldots, r$.

Proof

Proof. Let $y_i = x_i + 1$, then $y_i > 0$ and the number of non-negative solutions of

$$x_1 + x_2 + \cdots + x_r = n$$

is the same as the number of positive solutions of

$$(y_1 - 1) + (y_2 - 1) + \cdots + (y_r - 1) = n$$

i.e.,

$$y_1 + y_2 + \cdots + y_r = n + r,$$

which is $\binom{n+r-1}{r-1}$. □

1.6 Number of Integer Solutions

Example An investor has 20 thousand dollars to invest among 4 possible investments. Each investment must be in units of a thousand dollars. If the total 20 thousand is to be invested, how many different investment strategies are possible? What if not all the money need be invested?

If we let $x_i, i = 1, 2, 3, 4$, denote the number of thousands invested in investment i, then, when all is to be invested, x_1, x_2, x_3, x_4 are integers satisfying equation

$$x_1 + x_2 + x_3 + x_4 = 20, \quad x_i \geq 0$$

$$\binom{23}{3} = 1771$$

If not all of the money need be invested, then if we let x_5 denote the amount kept in reserve, a strategy is a nonnegative integer-valued vector x_1, x_2, x_3, x_4, x_5 satisfying the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

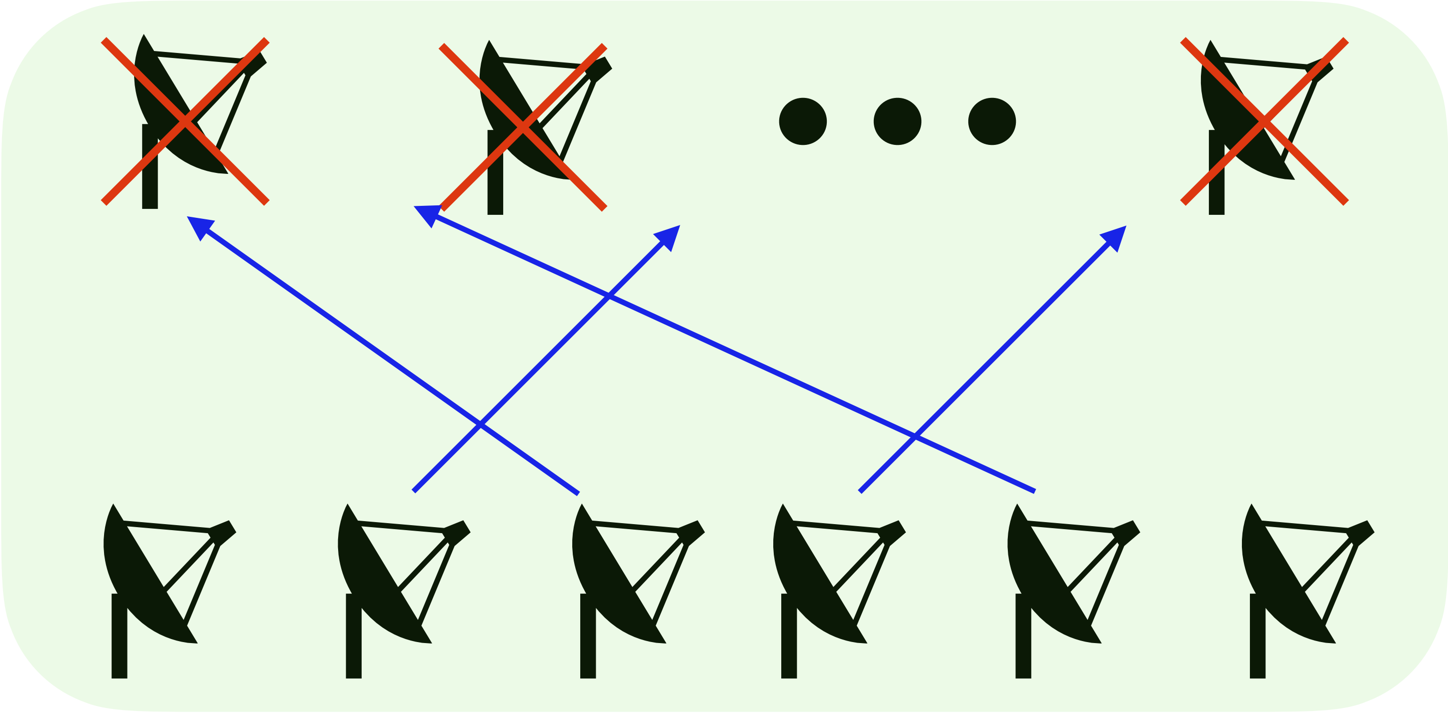
$$\binom{24}{4} = 10,626$$

1.6 Number of Integer Solutions

Example Consider a set of n antennas of which m are defective and $n - m$ are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?

How do we arrange them so that no two defectives are consecutive?

Fix the defective ones and insert functional ones into them



1.6 Number of Integer Solutions

Example Consider a set of n antennas of which m are defective and $n-m$ are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?

$$x_1 \quad 0 \quad x_2 \quad 0 \quad x_3 \quad 0 \quad \ldots \quad 0 \quad x_m \quad 0 \quad x_{m+1}$$

0 represents a defective antennas

x_1 *represents* x_1 *consecutive working antennas*

Now, there will be at least one functional item between any pair of defectives as long as $x_i > 0$, $i = 2, \ldots, m$. Hence, the number of outcomes satisfying the condition is the number of vectors $x_1, \ldots, x_{m+1}$ that satisfy the equation

$$x_1 + \cdots + x_{m+1} = n - m, \quad x_1 \geq 0, \quad x_{m+1} \geq 0, \quad x_i > 0, \quad i = 2, \ldots, m.$$

Let $y_1 = x_1 + 1$, $y_i = x_i$ for $i = 2, \ldots, m$ and $y_{m+1} = x_{m+1} + 1$, we see that this number is equal to the number of positive vectors $(y_1, \ldots, y_{m+1})$ that satisfy the equation

$$y_1 + y_2 + \cdots + y_{m+1} = n - m + 2.$$

there are $\binom{n-m+1}{m}$ such outcomes.